

## ON A CONJECTURE OF HONG AND WON

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ABSTRACT. We give an explicit counter-example to a conjecture of Kyusik Hong and Joonyeong Won about  $\alpha$ -invariants of polarized smooth del Pezzo surfaces of degree one.

All varieties are assumed to be algebraic, projective and defined over  $\mathbb{C}$ .

## 1. INTRODUCTION

In [10], Tian defined the  $\alpha$ -invariant of a smooth Fano variety and proved

**Theorem 1.1** ([10]). *Let  $X$  be a smooth Fano variety of dimension  $n$ . Suppose that*

$$\alpha(X) > \frac{n}{n+1}.$$

*Then  $X$  admits a Kähler–Einstein metric.*

Two-dimensional smooth Fano varieties are also known as smooth del Pezzo surfaces. The possible values of their  $\alpha$ -invariants are given by

**Theorem 1.2** ([1, Theorem 1.7]). *Let  $S$  be a smooth del Pezzo surface. Then*

$$\alpha(S) = \begin{cases} \frac{1}{3} & \text{if } S \cong \mathbb{F}_1 \text{ or } K_S^2 \in \{7, 9\}, \\ \frac{1}{2} & \text{if } S \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } K_S^2 \in \{5, 6\}, \\ \frac{2}{3} & \text{if } K_S^2 = 4, \\ \frac{2}{3} & \text{if } S \text{ is a cubic surface in } \mathbb{P}^3 \text{ with an Eckardt point,} \\ \frac{3}{4} & \text{if } S \text{ is a cubic surface in } \mathbb{P}^3 \text{ without Eckardt points,} \\ \frac{3}{4} & \text{if } K_S^2 = 2 \text{ and } |-K_S| \text{ has a tacnodal curve,} \\ \frac{5}{6} & \text{if } K_S^2 = 2 \text{ and } |-K_S| \text{ has no tacnodal curves,} \\ \frac{5}{6} & \text{if } K_S^2 = 1 \text{ and } |-K_S| \text{ has a cuspidal curve,} \\ 1 & \text{if } K_S^2 = 1 \text{ and } |-K_S| \text{ has no cuspidal curves.} \end{cases}$$

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*Key words and phrases.*  $\alpha$ -invariant of Tian, del Pezzo surface,  $K$ -stability.

Let  $X$  be an arbitrary smooth algebraic variety, and let  $L$  be an ample  $\mathbb{Q}$ -divisor on it. In [11], Tian defined a new invariant  $\alpha(X, L)$  that generalizes the classical  $\alpha$ -invariant. If  $X$  is a smooth Fano variety, then  $\alpha(X) = \alpha(X, -K_X)$ . By [3, Theorem A.3], one has

$$\alpha(X, L) = \sup \left\{ \lambda \in \mathbb{Q} \left| \begin{array}{l} \text{the log pair } (X, \lambda D) \text{ is log canonical} \\ \text{for every effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} L \end{array} \right. \right\} \in \mathbb{R}_{>0}.$$

In [8], Dervan proved

**Theorem 1.3** ([8, Theorem 1.1]). *Suppose that  $-K_X - \frac{n}{n+1} \frac{-K_X \cdot L^{n-1}}{L^n} L$  is nef, and*

$$\alpha(X, L) > \frac{n}{n+1} \frac{-K_X \cdot L^{n-1}}{L^n}.$$

*Then the pair  $(X, L)$  is  $K$ -stable.*

Donaldson, Tian and Yau conjectured that the following conditions are equivalent:

- the pair  $(X, L)$  is  $K$ -polystable,
- the variety  $X$  admits a constant scalar curvature Kähler metric in  $c_1(L)$ .

In [6], this conjecture has been proved in the case when  $X$  is a Fano variety and  $L = -K_X$ . Therefore, Dervan's Theorem 1.3 is a generalization of Tian's Theorem 1.1.

For smooth del Pezzo surfaces, Theorem 1.3 gives

**Theorem 1.4** ([9, 2]). *Let  $S$  be a smooth del Pezzo surface such that  $K_S^2 = 1$  or  $K_S^2 = 2$ . Let  $A$  be an ample  $\mathbb{Q}$ -divisor on the surface  $S$  such that the divisor*

$$-K_S - \frac{2 - K_S \cdot A}{3} \frac{A}{A^2}$$

*is nef. Then the pair  $(S, A)$  is  $K$ -stable.*

This result is closely related to

**Problem 1.5** (cf. Theorem 1.2). *Let  $S$  be a smooth del Pezzo surface. Compute*

$$\alpha(S, A) \in \mathbb{R}_{>0}$$

*for every ample  $\mathbb{Q}$ -divisor  $A$  on the surface  $S$ .*

Hong and Won suggested an answer to Problem 1.5 for del Pezzo surfaces of degree one. This answer is given by their [9, Conjecture 4.3], which is Conjecture 2.1 in Section 2.

The main result of this paper is

**Theorem 1.6** (cf. Theorem 1.2). *Let  $S$  be a smooth del Pezzo surface such that  $K_S^2 = 1$ . Let  $C$  be an irreducible smooth curve in  $S$  such that  $C^2 = -1$ . Then there is a unique curve*

$$\tilde{C} \in |-2K_S - C|.$$

*The curve  $\tilde{C}$  is also irreducible and smooth. One has  $\tilde{C}^2 = -1$  and  $1 \leq |C \cap \tilde{C}| \leq C \cdot \tilde{C} = 3$ . Let  $\lambda$  be a rational number such that  $0 \leq \lambda < 1$ . Then  $-K_S + \lambda C$  is ample and*

$$\alpha(S, -K_S + \lambda C) = \begin{cases} \min\left(\alpha(S), \frac{2}{1+2\lambda}\right) & \text{if } |C \cap \tilde{C}| \geq 2, \\ \min\left(\alpha(S), \frac{4}{3+3\lambda}\right) & \text{if } |C \cap \tilde{C}| = 1. \end{cases}$$

Theorem 1.6 implies that [9, Conjecture 4.3] is wrong. To be precise, this follows from

**Example 1.7.** Let  $S$  be a surface in  $\mathbb{P}(1, 1, 2, 3)$  that is given by

$$w^2 = z^3 + zx^2 + y^6,$$

where  $x, y, z, w$  are coordinates such that  $\text{wt}(x) = \text{wt}(y) = 1$ ,  $\text{wt}(z) = 2$  and  $\text{wt}(w) = 3$ . Then  $S$  is a smooth del Pezzo surface and  $K_S^2 = 1$ . Let  $C$  be the curve in  $X$  given by

$$z = w - y^3 = 0.$$

Similarly, let  $\tilde{C}$  be the curve in  $S$  that is given by  $z = w + y^3 = 0$ . Then  $C + \tilde{C} \sim -2K_S$ . Both curves  $C$  and  $\tilde{C}$  are smooth rational curves such that  $C^2 = \tilde{C}^2 = -1$  and  $|C \cap \tilde{C}| = 1$ . All singular curves in  $|-K_S|$  are nodal. Then  $\alpha(S) = 1$  by Theorems 1.2, so that

$$\alpha(S, -K_S + \lambda C) = \min\left(1, \frac{4}{3 + 3\lambda}\right)$$

by Theorem 1.6. But [9, Conjecture 4.3] says that  $\alpha(S, -K_S + \lambda C) = \min(1, \frac{2}{1+2\lambda})$ .

Theorem 1.6 has two applications. By Theorem 1.3, it implies

**Corollary 1.8** ([8, Theorem 1.2]). *Let  $S$  be a smooth del Pezzo surface such that  $K_S^2 = 1$ . Let  $C$  be an irreducible smooth curve in  $S$  such that  $C^2 = -1$ . Fix  $\lambda \in \mathbb{Q}$  such that*

$$3 - \sqrt{10} \leq \lambda \leq \frac{\sqrt{10} - 1}{9}.$$

*Then the pair  $(S, -K_S + \lambda C)$  is  $K$ -stable.*

By [5, Remark 1.1.3], Theorem 1.6 implies

**Corollary 1.9.** *Let  $S$  be a smooth del Pezzo surface. Suppose that  $K_S^2 = 1$  and  $\alpha(S) = 1$ . Let  $C$  be an irreducible smooth curve in  $S$  such that  $C^2 = -1$ . Fix  $\lambda \in \mathbb{Q}$  such that*

$$-\frac{1}{4} \leq \lambda \leq \frac{1}{3}.$$

*Then  $S$  does not contain  $(-K_S + \lambda C)$ -polar cylinders (see [5, Definition 1.2.1]).*

Corollary 1.8 follows from Theorem 1.4. Corollary 1.9 follows from [5, Theorem 2.2.3].

Let us describe the structure of this paper. In Section 2, we describe [9, Conjecture 4.3]. In Section 3, we present several well known local results about singularities of log pairs. In Section 4, we prove eight local lemmas that are crucial for the proof of Theorem 1.6. In Section 5, we prove Theorem 1.6 using Lemmas 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7, 4.8.

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## 2. CONJECTURE OF HONG AND WON

Let  $S$  be a smooth del Pezzo surface, and let  $A$  be an ample  $\mathbb{Q}$ -divisor on  $S$ . Put

$$\mu = \inf \left\{ \lambda \in \mathbb{Q}_{>0} \mid \text{the } \mathbb{Q}\text{-divisor } K_S + \lambda A \text{ is pseudo-effective} \right\} \in \mathbb{Q}_{>0}.$$

Then  $K_S + \mu A$  is contained in the boundary of the Mori cone  $\overline{\text{NE}}(S)$  of the surface  $S$ .

Suppose that  $K_S^2 = 1$ . Then  $\overline{\text{NE}}(S)$  is polyhedral and is generated by  $(-1)$ -curves in  $S$ . By a  $(-1)$ -curve, we mean a smooth irreducible rational curve  $E \subset S$  such that  $E^2 = -1$ .

Let  $\Delta_A$  be the smallest extremal face of the Mori cone  $\overline{\text{NE}}(S)$  that contains  $K_S + \mu A$ . Let  $\phi: S \rightarrow Z$  be the contraction given by the face  $\Delta_A$ . Then

- either  $\phi$  is a birational morphism and  $Z$  is a smooth del Pezzo surface,
- or  $\phi$  is a conic bundle and  $Z \cong \mathbb{P}^1$ .

If  $\phi$  is birational and  $Z \not\cong \mathbb{P}^1 \times \mathbb{P}^1$ , we call  $A$  a divisor of  $\mathbb{P}^2$ -type. In this case, we have

$$K_S + \mu A \sim_{\mathbb{Q}} \sum_{i=1}^8 a_i E_i,$$

where  $E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8$  are eight disjoint  $(-1)$ -curves in our surface  $S$ , and  $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$  are non-negative rational numbers such that

$$1 > a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \geq a_7 \geq a_8 \geq 0.$$

In this case, we put  $s_A = a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8$ .

If our ample divisor  $A$  is not a divisor of  $\mathbb{P}^2$ -type, then the surface  $S$  contains a smooth irreducible rational curve  $C$  such that  $C^2 = 0$  and

$$K_S + \mu A \sim_{\mathbb{Q}} \delta C + \sum_{i=1}^7 a_i E_i,$$

where  $E_1, E_2, E_3, E_4, E_5, E_6, E_7$  are disjoint  $(-1)$ -curves in  $S$  that are disjoint from  $C$ , and  $\delta, a_1, a_2, a_3, a_4, a_5, a_6, a_7$  are non-negative rational numbers such that

$$1 > a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \geq a_7 \geq 0.$$

In this case, let  $\psi: S \rightarrow \overline{S}$  be the contraction of the curves  $E_1, E_2, E_3, E_4, E_5, E_6, E_7$ , and let  $\eta: S \rightarrow \mathbb{P}^1$  be a conic bundle given by  $|C|$ . Then either  $\overline{S} \cong \mathbb{F}_1$  or  $\overline{S} \cong \mathbb{P}^1 \times \mathbb{P}^1$ . In both cases, there exists a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\psi} & \overline{S} \\ & \searrow \eta & \swarrow \pi \\ & \mathbb{P}^1 & \end{array}$$

where  $\pi$  is a natural projection. Then  $\delta > 0 \iff \phi$  is a conic bundle and  $\phi = \eta$ . Similarly, if  $\phi$  is birational and  $Z \cong \mathbb{P}^1 \times \mathbb{P}^1$ , then  $\delta = 0$ ,  $a_7 > 0$ , and  $\phi = \psi$ . Then

- we call  $A$  a divisor of  $\mathbb{F}_1$ -type in the case when  $\overline{S} \cong \mathbb{F}_1$ ,
- we call  $A$  a divisor of  $\mathbb{P}^1 \times \mathbb{P}^1$ -type in the case when  $\overline{S} \cong \mathbb{P}^1 \times \mathbb{P}^1$ .

In both cases, we put  $s_A = a_2 + a_3 + a_4 + a_5 + a_6 + a_7$ .

In order to study  $\alpha(S, A)$ , we may assume that  $\mu = 1$ , because

$$\alpha(S, A) = \frac{\alpha(S, \mu A)}{\mu}.$$

If  $A$  is a divisor of  $\mathbb{P}^2$ -type, let us define a number  $\alpha_c(S, A)$  as follows:

- if  $s_A > 4$ , we put  $\alpha_c(S, A) = \frac{1}{2+a_1}$ ,
- if  $4 \geq s_A > 1$ , we let  $\alpha_c(S, A)$  to be

$$\max\left(\frac{2}{2+2a_1+s_A-a_2-a_3}, \frac{4}{3+4a_1+2s_A-a_2-a_3-a_4}, \frac{3}{2+3a_1+s_A}\right),$$

- if  $1 \geq s_A$ , we put  $\alpha_c(S, A) = \min(\frac{2}{1+2a_1+s_A}, 1)$ .

Similarly, if  $A$  is a divisor of  $\mathbb{F}_1$ -type, we define  $\alpha_c(S, A)$  as follows:

- if  $s_A > 4$ , we put  $\alpha_c(S, A) = \frac{1}{2+a_1+\delta}$ ,
- if  $4 \geq s_A > 1$ , we let  $\alpha_c(S, A)$  to be

$$\max\left(\frac{2}{2+2a_1+s_A-a_2-a_3+2\delta}, \frac{4}{3+4a_1+2s_A-a_2-a_3-a_4+4\delta}, \frac{3}{2+3a_1+s_A+3\delta}\right),$$

- if  $1 \geq s_A$ , we put  $\alpha_c(S, A) = \min(\frac{2}{1+2a_1+s_A+2\delta}, 1)$ .

Finally, if  $A$  is a divisor of  $\mathbb{P}^1 \times \mathbb{P}^1$ -type, we define  $\alpha_c(S, A)$  as follows:

- if  $s_A > 4$ , we put  $\alpha_c(S, A) = \frac{1}{2+a_1+\delta}$ ,
- if  $4 \geq s_A > 1$ , we let  $\alpha_c(S, A)$  to be

$$\max\left(\frac{2}{2+s_A-a_7-a_2-a_3+2\delta}, \frac{4}{3+2s_A-2a_7-a_2-a_3-a_4+4\delta}, \frac{3}{2+s_A-a_7+3\delta}\right),$$

- if  $1 \geq s_A$ , we put  $\alpha_c(S, A) = \min(\frac{2}{1+s_A-a_7+2\delta}, 1)$ .

The conjecture of Hong and Won is

**Conjecture 2.1** ([9, Conjecture 4.3]). *If  $\alpha(S) = 1$ , then  $\alpha(S, A) = \alpha_c(S, A)$ .*

The main evidence for this conjecture is

**Theorem 2.2** ([9]). *Let  $D$  be an effective  $\mathbb{Q}$ -divisor on the surface  $S$  such that  $D \sim_{\mathbb{Q}} A$ . Then the log pair  $(S, \alpha_c(S, A)D)$  is log canonical outside of finitely many points.*

As we already mentioned in Section 1, Example 1.7 shows that Conjecture 2.1 is wrong. However, the smooth del Pezzo surface of degree one in Example 1.7 is rather special. Therefore, Conjecture 2.1 may hold for *general* smooth del Pezzo surfaces of degree one.

By [5, Remark 1.1.3], it follows from Conjecture 2.1 that  $S$  does not contain  $A$ -polar cylinders (see [5, Definition 1.2.1]) when  $\alpha(S) = 1$  and  $a_1$  and  $\delta$  are small enough.

## 3. SINGULARITIES OF LOG PAIRS

Let  $S$  be a smooth surface, and let  $D$  be an effective  $\mathbb{Q}$ -divisor on it. Write

$$D = \sum_{i=1}^r a_i C_i$$

where each  $C_i$  is an irreducible curve on  $S$ , and each  $a_i$  is a non-negative rational number. We assume here that all curves  $C_1, \dots, C_r$  are different.

Let  $\gamma: \mathcal{S} \rightarrow S$  be a birational morphism such that the surface  $\mathcal{S}$  is smooth as well. It is well-known that the morphism  $\gamma$  is a composition of  $n$  blow ups of smooth points. Thus, the morphism  $\gamma$  contracts  $n$  irreducible curves. Denote these curves by  $\Gamma_1, \dots, \Gamma_n$ . For each curve  $C_i$ , denote by  $\mathcal{C}_i$  its proper transform on the surface  $\mathcal{S}$ . Then

$$K_{\mathcal{S}} + \sum_{i=1}^r a_i \mathcal{C}_i + \sum_{j=1}^n b_j \Gamma_j \sim_{\mathbb{Q}} \gamma^*(K_S + D)$$

for some rational numbers  $b_1, \dots, b_n$ . Suppose, in addition, that the divisor

$$\sum_{i=1}^r \mathcal{C}_i + \sum_{j=1}^n \Gamma_j$$

has simple normal crossing singularities. Fix a point  $P \in S$ .

**Definition 3.1.** The log pair  $(S, D)$  is *log canonical* (respectively *Kawamata log terminal*) at the point  $P$  if the following two conditions are satisfied:

- $a_i \leq 1$  (respectively  $a_i < 1$ ) for every  $C_i$  such that  $P \in C_i$ ,
- $b_j \leq 1$  (respectively  $b_j < 1$ ) for every  $\Gamma_j$  such that  $\pi(\Gamma_j) = P$ .

This definition does not depend on the choice of the birational morphism  $\gamma$ .

The log pair  $(S, D)$  is said to be *log canonical* (respectively *Kawamata log terminal*) if it is log canonical (respectively, *Kawamata log terminal*) at every point in  $S$ .

The following result follows from Definition 3.1. But it is very handy.

**Lemma 3.2.** *Suppose that the singularities of the pair  $(S, D)$  are not log canonical at  $P$ . Let  $D'$  be an effective  $\mathbb{Q}$ -divisor on  $S$  such that  $(S, D')$  is log canonical at  $P$  and  $D' \sim_{\mathbb{Q}} D$ . Then there exists an effective  $\mathbb{Q}$ -divisor  $D''$  on the surface  $S$  such that*

$$D'' \sim_{\mathbb{Q}} D,$$

*the log pair  $(S, D'')$  is not log canonical at  $P$ , and  $\text{Supp}(D') \not\subseteq \text{Supp}(D'')$ .*

*Proof.* Let  $\epsilon$  be the largest rational number such that  $(1 + \epsilon)D - \epsilon D'$  is effective. Then

$$(1 + \epsilon)D - \epsilon D' \sim_{\mathbb{Q}} D.$$

Put  $D'' = (1 + \epsilon)D - \epsilon D'$ . Then  $(S, D'')$  is not log canonical at  $P$ , because

$$D = \frac{1}{1 + \epsilon} D'' + \frac{\epsilon}{1 + \epsilon} D'.$$

Furthermore, we have  $\text{Supp}(D') \not\subseteq \text{Supp}(D'')$  by construction. □

Let  $f: \tilde{S} \rightarrow S$  be a blow up of the point  $P$ . Let us denote the  $f$ -exceptional curve by  $F$ . Denote by  $\tilde{D}$  the proper transform of the divisor  $D$  via  $f$ . Put  $m = \text{mult}_P(D)$ .

**Theorem 3.3** ([7, Exercise 6.18]). *If  $(S, D)$  is not log canonical at  $P$ , then  $m > 1$ .*

Let  $C$  be an irreducible curve in the surface  $S$ . Suppose that  $P \in C$  and  $C \not\subseteq \text{Supp}(D)$ . Denote by  $\tilde{C}$  the proper transform of the curve  $C$  via  $f$ . Fix  $a \in \mathbb{Q}$  such that  $0 \leq a \leq 1$ . Then  $(S, aC + D)$  is not log canonical at  $P$  if and only if the log pair

$$(3.1) \quad \left( \tilde{S}, a\tilde{C} + \tilde{D} + (a\text{mult}_P(C) + m - 1)F \right)$$

is not log canonical at some point in  $F$ . This follows from Definition 3.1.

**Theorem 3.4** ([7, Exercise 6.31]). *Suppose that  $C$  is smooth at  $P$ , and  $(D \cdot C)_P \leq 1$ . Then the log pair  $(S, aC + D)$  is log canonical at  $P$ .*

**Corollary 3.5.** *Suppose that the log pair (3.1) is not log canonical at some point in  $F \setminus \tilde{C}$ . Then either  $a\text{mult}_P(C) + m > 2$  or  $m > 1$  (or both).*

Let us give another application of Theorem 3.4.

**Lemma 3.6.** *Suppose that there is a double cover  $\pi: S \rightarrow \mathbb{P}^2$  branched in a curve  $R \subset \mathbb{P}^2$ . Suppose also that  $(S, D)$  is not log canonical at  $P$ , and  $D \sim_{\mathbb{Q}} \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ . Then  $\pi(P) \in R$ .*

*Proof.* The log pair  $(\tilde{S}, \tilde{D} + (m - 1)F)$  is not log canonical at some point  $Q \in F$ . Then

$$(3.2) \quad m + \text{mult}_Q(\tilde{D}) > 2$$

by Theorem 3.3. Suppose that  $\pi(P) \notin R$ . Then there is  $Z \in |\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))|$  such that

- the curve  $Z$  passes through the point  $P$ ,
- the proper transform of the curve  $Z$  on the surface  $\tilde{S}$  contains  $Q$ .

Denote by  $\tilde{Z}$  the proper transform of the curve  $Z$  on the surface  $\tilde{S}$ .

By Lemma 3.2, we may assume that the support of the  $\mathbb{Q}$ -divisor  $D$  does not contain at least one irreducible component of the curve  $Z$ , because  $(S, Z)$  is log canonical at  $P$ . Thus, if  $Z$  is irreducible, then  $2 - m = \tilde{Z} \cdot \tilde{D} \geq \text{mult}_Q(\tilde{D})$ , which contradicts (3.2).

We see that  $Z = Z_1 + Z_2$ , where  $Z_1$  and  $Z_2$  are irreducible smooth rational curves. We may assume that  $Z_2 \not\subseteq \text{Supp}(D)$ . If  $P \in Z_2$ , then  $1 = D \cdot Z_2 \geq m > 1$  by Theorem 3.3. This shows that  $P \in Z_1$  and  $Z_1 \subseteq \text{Supp}(D)$ .

Let  $d$  be the degree of the curve  $R$ . Then  $Z_1^2 = Z_2^2 = \frac{2-d}{2}$  and  $Z_1 \cdot Z_2 = \frac{d}{2}$ .

We may assume that  $C_1 = Z_1$ . Put  $\Delta = a_2 C_2 + \cdots + a_r C_r$ . Then  $a_1 \leq \frac{2}{d}$ , since

$$1 = Z_2 \cdot D = Z_2 \cdot (a_1 C_1 + \Delta) = a_1 Z_2 \cdot C_1 + Z_2 \cdot \Delta \geq a_1 Z_2 \cdot C_1 = \frac{a_1 d}{2}.$$

Denote by  $\tilde{C}_1$  the proper transform of the curve  $C_1$  on the surface  $\tilde{S}$ . Then  $Q \in \tilde{C}_1$ . Denote by  $\tilde{\Delta}$  the proper transform of the  $\mathbb{Q}$ -divisor  $\Delta$  on the surface  $\tilde{S}$ . The log pair

$$\left( \tilde{S}, a_1 \tilde{C}_1 + \tilde{\Delta} + (a_1 + \text{mult}_P(\Delta) - 1)F \right)$$

is not log canonical at the point  $Q$  by construction. By Theorem 3.4, we have

$$1 + \frac{d-2}{2}a_1 - \text{mult}_P(\Delta) = \tilde{C}_1 \cdot \tilde{\Delta} \geq (\tilde{C}_1 \cdot \tilde{\Delta})_Q > 1 - (a_1 + \text{mult}_P(\Delta) - 1),$$

so that  $a_1 > \frac{2}{d}$ . But we already proved that  $a_1 \leq \frac{2}{d}$ .  $\square$

Fix a point  $Q \in F$ . Put  $\tilde{m} = \text{mult}_Q(\tilde{D})$ . Let  $g: \hat{S} \rightarrow \tilde{S}$  be a blow up of the point  $Q$ . Denote by  $\hat{C}$  and  $\hat{F}$  the proper transforms of the curves  $\tilde{C}$  and  $F$  via  $g$ , respectively. Similarly, let us denote by  $\hat{D}$  the proper transform of the  $\mathbb{Q}$ -divisor  $D$  on the surface  $\hat{S}$ . Denote by  $G$  the  $g$ -exceptional curve. If the log pair (3.1) is not log canonical at  $Q$ , then

$$(3.3) \quad \left( \hat{S}, a\hat{C} + \hat{D} + (a\text{mult}_P(C) + m - 1)\hat{F} + (a\text{mult}_P(C) + a\text{mult}_Q(\tilde{C}) + m + \tilde{m} - 2)G \right)$$

is not log canonical at some point in  $G$ .

**Lemma 3.7.** *Suppose  $m \leq 1$ ,  $a\text{mult}_P(C) + m \leq 2$  and  $a\text{mult}_P(C) + a\text{mult}_Q(\tilde{C}) + 2m \leq 3$ . Then (3.3) is log canonical at every point in  $G \setminus \hat{C}$ .*

*Proof.* Suppose that (3.3) is not log canonical at some point  $O \in G$  such that  $O \notin \hat{C}$ . If  $O \notin \hat{F}$ , then  $1 \geq m \geq \tilde{m} = \hat{D} \cdot G \geq (\hat{D} \cdot G)_O > 1$  by Theorem 3.4. Then  $O \in \hat{F}$ . Then

$$m - \tilde{m} = (\hat{D} \cdot \hat{F})_O > 1 - (a\text{mult}_P(C) + a\text{mult}_Q(\tilde{C}) + m + \tilde{m} - 2)$$

by Theorem 3.4. This is impossible, since  $a\text{mult}_P(C) + a\text{mult}_Q(\tilde{C}) + 2m \leq 3$ .  $\square$

Fix a point  $O \in G$ . Put  $\hat{m} = \text{mult}_O(\hat{D})$ . Let  $h: \bar{S} \rightarrow \hat{S}$  be a blow up of the point  $O$ . Denote by  $\bar{C}$ ,  $\bar{F}$ ,  $\bar{G}$  the proper transforms of the curves  $\hat{C}$ ,  $\hat{F}$  and  $G$  via  $h$ , respectively. Similarly, let us denote by  $\bar{D}$  the proper transform of the  $\mathbb{Q}$ -divisor  $D$  on the surface  $\bar{S}$ . Let  $H$  be the  $h$ -exceptional curve. If  $O = G \cap \hat{F}$  and (3.3) is not log canonical at  $O$ , then

$$(3.4) \quad \left( \bar{S}, a\bar{C} + \bar{D} + (2a\text{mult}_P(C) + a\text{mult}_Q(\tilde{C}) + a\text{mult}_O(\hat{C}) + 2m + \tilde{m} + \hat{m} - 4)H + \right. \\ \left. + (a\text{mult}_P(C) + m - 1)\bar{F} + (a\text{mult}_P(C) + a\text{mult}_Q(\tilde{C}) + m + \tilde{m} - 2)\bar{G} \right)$$

is not log canonical at some point in  $H$ .

**Lemma 3.8.** *Suppose that  $O = G \cap \hat{F}$ ,  $m \leq 1$ ,  $a\text{mult}_P(C) + a\text{mult}_Q(\tilde{C}) + m + \tilde{m} \leq 3$  and*

$$2a\text{mult}_P(C) + a\text{mult}_Q(\tilde{C}) + a\text{mult}_O(\hat{C}) + 4m \leq 5.$$

*Then the log pair (3.4) is log canonical at every point in  $H \setminus \bar{C}$ .*

*Proof.* Suppose that the pair (3.4) is not log canonical at some  $E \in H$  such that  $E \notin \bar{C}$ . If  $E \notin \bar{F} \cup \bar{G}$ , then  $m \geq \hat{m} = \bar{D} \cdot H \geq (\bar{D} \cdot H)_E > 1$  by Theorem 3.4. Then  $E \in \bar{F} \cup \bar{G}$ .

If  $E \in \bar{G}$ , then  $E \notin \bar{F}$ , so that Theorem 3.4 gives

$$\tilde{m} - \hat{m} = (\bar{D} \cdot \bar{F})_E > 1 - (2a\text{mult}_P(C) + a\text{mult}_Q(\tilde{C}) + a\text{mult}_O(\hat{C}) + 2m + \tilde{m} + \hat{m} - 4),$$



which is impossible, since  $2\text{mult}_P(C) + \text{mult}_Q(\tilde{C}) + \text{mult}_O(\hat{C}) + 4m \leq 5$  by assumption. Similarly, if  $E \in \overline{F}$ , then  $E \notin \overline{G}$ , so that Theorem 3.4 gives

$$m - \tilde{m} - \hat{m} = (\overline{D} \cdot \overline{F})_E > 1 - \left( 2\text{mult}_P(C) + \text{mult}_Q(\tilde{C}) + \text{mult}_O(\hat{C}) + 2m + \tilde{m} + \hat{m} - 4 \right),$$

which is impossible, since  $2\text{mult}_P(C) + \text{mult}_Q(\tilde{C}) + \text{mult}_O(\hat{C}) + 4m \leq 5$ .  $\square$

Let  $Z$  be an irreducible curve in  $S$  such that  $P \in Z$ . Suppose also that  $Z \not\subseteq \text{Supp}(D)$ . Denote its proper transforms on the surfaces  $\tilde{S}$  and  $\hat{S}$  by the symbols  $\tilde{Z}$  and  $\hat{Z}$ , respectively. Fix  $b \in \mathbb{Q}$  such that  $0 \leq b \leq 1$ . If  $(S, aC + bZ + D)$  is not log canonical at  $P$ , then

$$(3.5) \quad \left( \tilde{S}, a\tilde{C} + b\tilde{Z} + \tilde{D} + \left( \text{mult}_P(C) + b\text{mult}_P(Z) + m - 1 \right) F \right)$$

is not log canonical at some point in  $F$ .

**Lemma 3.9.** *Suppose that  $m \leq 1$  and*

$$\text{mult}_P(C) + b\text{mult}_P(Z) + m \leq 2.$$

*Then (3.5) is log canonical at every point in  $Q \in F \setminus (\tilde{C} \cup \tilde{Z})$ .*

*Proof.* Suppose that (3.5) is not log canonical at some point  $Q \in F$  such that  $Q \notin \tilde{C} \cup \tilde{Z}$ . Then  $m = \tilde{D} \cdot F \geq (\tilde{D} \cdot F)_Q > 1$  by Theorem 3.4. But  $m \leq 1$  by assumption.  $\square$

If the log pair (3.5) is not log canonical at  $Q$ , then the log pair

$$(3.6) \quad \left( \hat{S}, a\hat{C} + b\hat{Z} + \hat{D} + \left( \text{mult}_P(C) + b\text{mult}_P(Z) + m - 1 \right) F + \right. \\ \left. + \left( \text{mult}_P(C) + \text{mult}_Q(\tilde{C}) + b\text{mult}_P(Z) + b\text{mult}_Q(\tilde{Z}) + m + \tilde{m} - 2 \right) G \right)$$

is not log canonical at some point in  $G$ .

**Lemma 3.10.** *Suppose that  $m \leq 1$ ,  $\text{mult}_P(C) + b\text{mult}_P(Z) + m \leq 2$  and*

$$\text{mult}_P(C) + \text{mult}_Q(\tilde{C}) + b\text{mult}_P(Z) + b\text{mult}_Q(\tilde{Z}) + 2m \leq 3.$$

*Then the log pair (3.6) is log canonical at every point in  $G \setminus (\hat{C} \cup \hat{Z})$ .*

*Proof.* We may assume that the log pair (3.6) is not log canonical at  $O$  and  $O \notin \hat{C} \cup \hat{Z}$ . If  $O \notin \hat{F}$ , then  $m \geq \tilde{m} = \hat{D} \cdot G \geq (\hat{D} \cdot G)_O > 1$  by Theorem 3.4, so that  $O \in \hat{F}$ . Then

$$m - \tilde{m} = (\hat{D} \cdot \hat{F})_O > 1 - \left( \text{mult}_P(C) + \text{mult}_Q(\tilde{C}) + b\text{mult}_P(Z) + b\text{mult}_Q(\tilde{Z}) + m + \tilde{m} - 2 \right),$$

by Theorem 3.4, so that  $\text{mult}_P(C) + \text{mult}_Q(\tilde{C}) + b\text{mult}_P(Z) + b\text{mult}_Q(\tilde{Z}) + 2m > 3$ .  $\square$

## 4. EIGHT LOCAL LEMMAS

Let us use notations and assumptions of Section 3. Fix  $x \in \mathbb{Q}$  such that  $0 \leq x \leq 1$ . Put

$$\text{lct}_P(S, C) = \sup \left\{ \lambda \in \mathbb{Q} \mid \text{the log pair } (S, \lambda C) \text{ is log canonical at } P \right\} \in \mathbb{Q}_{>0}.$$

**Lemma 4.1.** *Suppose that  $C$  has an ordinary node or an ordinary cusp at  $P$ ,  $a \leq \frac{x}{2}$  and*

$$(D \cdot C)_P \leq \frac{4}{3} + \frac{x}{6} - a.$$

*Then the log pair  $(S, aC + D)$  is log canonical at  $P$ .*

*Proof.* We have  $2m \leq \text{mult}_P(D)\text{mult}_P(C) \leq (D \cdot C)_P \leq \frac{4}{3} + \frac{x}{6} - a$ , so that  $2m + a \leq \frac{4}{3} + \frac{x}{6}$ . Then  $m \leq \frac{3}{4}$  and  $m + 2a = m + \frac{a}{2} + \frac{3a}{2} \leq \frac{\frac{4}{3} + \frac{x}{6}}{2} + \frac{3a}{2} \leq \frac{\frac{4}{3} + \frac{x}{6}}{2} + \frac{3x}{4} = \frac{2}{3} + \frac{5}{6}x \leq \frac{3}{2}$ .

Suppose that  $(S, aC + D)$  is not log canonical at  $P$ . Let us seek for a contradiction. We may assume that (3.1) is not log canonical at  $Q$ . Then  $Q \in \tilde{C}$  by Corollary 3.5. Then

$$(\tilde{D} \cdot \tilde{C})_O > 1 - (2a + m - 1)(\tilde{C} \cdot F)_O \geq 1 - 2(2a + m - 1) = 3 - 4a - 2m.$$

On the other hand, we have  $\frac{4}{3} + \frac{x}{6} - a \geq (D \cdot C)_P \geq 2m + (\tilde{D} \cdot \tilde{C})_O$ , so that  $a > \frac{5}{9} - \frac{x}{18}$ . Then  $\frac{x}{2} \geq a > \frac{5}{9} - \frac{x}{18}$ , so that  $x > 1$ . But  $x \leq 1$  by assumption.  $\square$

**Lemma 4.2.** *Suppose that  $C$  has an ordinary node or an ordinary cusp at  $P$ , and*

$$(D \cdot C)_P \leq \text{lct}_P(S, C) + \frac{x}{2}.$$

*Suppose also that  $a \leq \text{lct}_P(S, C) - \frac{x}{2}$ . Then  $(S, aC + D)$  is log canonical at  $P$ .*

*Proof.* We have  $2m \leq (D \cdot C)_P$ . This gives  $2m + a \leq 1 + \frac{x}{2}$ . Thus, we have  $m \leq \frac{1+\frac{x}{2}}{2} \leq \frac{3}{4}$ . Similarly, we get  $m + 2a = m + \frac{a}{2} + \frac{3a}{2} \leq \frac{1+\frac{x}{2}}{2} + \frac{3a}{2} \leq \frac{1+\frac{x}{2}}{2} + \frac{3}{2}(1 - \frac{x}{2}) = 2 - \frac{x}{2} \leq 2$ .

Suppose that  $(S, aC + D)$  is not log canonical at  $P$ . Let us seek for a contradiction. We may assume that the pair (3.1) is not log canonical at  $Q$ . Then  $Q \in \tilde{C}$  by Corollary 3.5. We may assume that (3.3) is not log canonical at  $O$ . Then  $O \in \hat{C}$  by Lemma 3.7, since

$$3a + 2m \leq 2a + 1 + \frac{x}{2} \leq 2 - x + 1 + \frac{x}{2} = 3 - \frac{x}{2} \leq 3,$$

because  $2m + a \leq 1 + \frac{x}{2}$  and  $a \leq 1 - \frac{x}{2}$ . If  $O \notin \hat{F}$ , then Theorem 3.4 gives

$$1 + \frac{x}{2} - a \geq (D \cdot C)_P - 2m - \tilde{m} \geq (\hat{D} \cdot \hat{C})_O > 1 - (3a + m + \tilde{m} - 2),$$

which implies that  $2a + \frac{x}{2} > 2 + m$ . But  $2a + \frac{x}{2} \leq 2 - \frac{x}{2}$ , because  $a \leq \text{lct}_P(S, C) - \frac{x}{2} \leq 1 - \frac{x}{2}$ . This shows that  $O = G \cap \hat{F} \cap \hat{C}$ . In particular, the curve  $C$  has an ordinary cusp at  $P$ . By assumption, we have  $a \leq \frac{5}{6} - \frac{x}{2}$  and  $2m + a \leq \frac{5}{6} + \frac{x}{2}$ . This gives  $6a + 4m \leq 5 - x \leq 5$ .

Put  $E = H \cap \bar{C}$ . Then (3.4) is not log canonical at  $E$  by Lemma 3.8. Then

$$(\bar{D} \cdot \bar{C})_E > 1 - (6a + 2m + \tilde{m} + \hat{m} - 4) = 5 - 6a - 2m - \tilde{m} - \hat{m}$$

by Theorem 3.4. Thus, we have  $\frac{5}{6} + \frac{x}{2} - a \geq (D \cdot C)_P \geq 2m + \tilde{m} + \hat{m} + (\bar{D} \cdot \bar{C})_E > 5 - 6a$ . This gives  $a > \frac{5}{6} - \frac{x}{10}$ . But  $a \leq \frac{5}{6} - \frac{x}{2}$ , which is absurd.  $\square$

**Lemma 4.3.** *Suppose that  $C$  is smooth at  $P$ ,  $a \leq \frac{1}{3} + \frac{x}{2}$ ,  $m + a \leq 1 + \frac{x}{2}$  and*

$$(D \cdot C)_P \leq 1 - \frac{x}{2} + a.$$

*Then the log pair  $(S, aC + D)$  is log canonical at  $P$ .*

*Proof.* We have  $m \leq (D \cdot C)_P$ , so that  $m - a \leq 1 - \frac{x}{2}$ . Then  $m \leq 1$ , since  $m + a \leq 1 + \frac{x}{2}$ .

Suppose that  $(S, aC + D)$  is not log canonical at  $P$ . Let us seek for a contradiction. We may assume that the pair (3.1) is not log canonical at  $Q$ . Then  $Q \in \tilde{C}$  by Corollary 3.5. We may assume that (3.3) is not log canonical at  $O$ . Then  $O \in \hat{C}$  by Lemmas 3.7. Then

$$(\hat{D} \cdot \hat{C})_O > 1 - (2a + m + \tilde{m} - 2) = 3 - 2a - m - \tilde{m}$$

by Theorem 3.4. Then  $1 - \frac{x}{2} + a \geq (D \cdot C)_P \geq m + (\tilde{D} \cdot \tilde{C})_Q \geq m + \tilde{m} + (\hat{D} \cdot \hat{C})_O > 3 - 2a$ . This gives  $a > \frac{2}{3} + \frac{x}{6}$ , which is impossible, since  $a \leq \frac{1}{3} + \frac{x}{2}$  and  $x \leq 1$ .  $\square$

**Lemma 4.4.** *Suppose that  $C$  is smooth at  $P$ ,  $a \leq \frac{8}{9} - \frac{x}{18}$ ,  $m + a \leq \frac{4}{3} + \frac{x}{6}$  and*

$$(D \cdot C)_P \leq \frac{x}{2} + a.$$

*Then the log pair  $(S, aC + D)$  is log canonical at  $P$ .*

*Proof.* We have  $m \leq (D \cdot C)_P$ , so that  $m - a \leq \frac{x}{2}$ . Then  $m \leq \frac{2}{3} + \frac{x}{3} \leq 1$ , since  $m + a \leq \frac{4}{3} + \frac{x}{6}$ .

Suppose that  $(S, aC + D)$  is not log canonical at  $P$ . Let us seek for a contradiction. We may assume that the pair (3.1) is not log canonical at  $Q$ . Then  $Q \in \tilde{C}$  by Corollary 3.5. We may assume that (3.3) is not log canonical at  $O$ . Then  $O \in \hat{C}$  by Lemmas 3.7. Then

$$(\hat{D} \cdot \hat{C})_O > 1 - (2a + m + \tilde{m} - 2) = 3 - 2a - m - \tilde{m}$$

by Theorem 3.4. Then  $\frac{x}{2} + a \geq (D \cdot C)_P \geq m + (\tilde{D} \cdot \tilde{C})_Q \geq m + \tilde{m} + (\hat{D} \cdot \hat{C})_O > 3 - 2a$ . This gives  $a > 1 - \frac{x}{6}$ , which is impossible, since  $a \leq \frac{8}{9} - \frac{x}{18}$  and  $x \leq 1$ .  $\square$

**Lemma 4.5.** *Suppose that  $C$  has an ordinary node or an ordinary cusp at  $P$ ,  $a \leq \frac{1+x}{3}$  and*

$$(D \cdot C)_P \leq 2 - 2a.$$

*Then the log pair  $(S, aC + D)$  is log canonical at  $P$ .*

*Proof.* We have  $2m \leq (D \cdot C)_P \leq 2 - 2a$ . This gives  $m + a \leq 1$ , so that we have  $m \leq 1$ . Then  $m + 2a \leq 1 + a \leq 1 + \frac{1+x}{3} = \frac{4+x}{3} \leq \frac{5}{3}$  and  $3a + 2m \leq 2 + a \leq 2 + \frac{1+x}{3} = \frac{7+x}{3} \leq \frac{8}{3}$ .

Suppose that  $(S, aC + D)$  is not log canonical at  $P$ . Let us seek for a contradiction. We may assume that the pair (3.1) is not log canonical at  $Q$ . Then  $Q \in \tilde{C}$  by Corollary 3.5. We may assume that (3.3) is not log canonical at  $O$ . Then  $O \in \hat{C}$  by Lemma 3.7.

If  $O \notin \hat{F}$ , then  $(\hat{D} \cdot \hat{C})_O > 3 - 3a - m - \tilde{m}$  by Theorem 3.4, so that

$$2 - 2a \geq (D \cdot C)_P \geq 2m + (\tilde{D} \cdot \tilde{C})_Q \geq 2m + \tilde{m} + (\hat{D} \cdot \hat{C})_O > 3 - 3a,$$

which is absurd. This shows that  $O = G \cap \hat{F} \cap \hat{C}$ . Then

$$(\hat{D} \cdot \hat{C})_O > 1 - (2a + m - 1) - (3a + m + \tilde{m} - 2) = 4 - 5a - 2m - \tilde{m}$$

by Theorem 3.4. Then  $2 - 2a \geq (D \cdot C)_P \geq 2m + \tilde{m} + (\hat{D} \cdot \hat{C})_O > 4 - 5a$ , so that  $a > \frac{2}{3}$ . But  $a \leq \frac{1+x}{3} \leq \frac{2}{3}$  by assumption. This is a contradiction.  $\square$

**Lemma 4.6.** *Suppose that  $C$  has an ordinary node or an ordinary cusp at  $P$ ,  $a \leq \frac{2}{3}$  and*

$$(D \cdot C)_P \leq \frac{4}{3} + \frac{2x}{3} - 2a.$$

*Then the log pair  $(S, aC + D)$  is log canonical at  $P$ .*

*Proof.* We have  $2m \leq (D \cdot C)_P$ , so that  $m + a \leq \frac{2}{3} + \frac{x}{3} \leq 1$ . Then  $m \leq 1$  and  $m + 2a \leq \frac{5}{3}$ . Similarly, we see that  $3a + 2m \leq \frac{4}{3} + \frac{2x}{3} + a \leq \frac{4}{3} + \frac{2x}{3} + \frac{2}{3} = 2 + \frac{2x}{3} \leq \frac{8}{3} < 3$ .

Suppose that  $(S, aC + D)$  is not log canonical at  $P$ . Let us seek for a contradiction. We may assume that the pair (3.1) is not log canonical at  $Q$ . Then  $Q \in \tilde{C}$  by Corollary 3.5. We may assume that (3.3) is not log canonical at  $O$ . Then  $O \in \hat{C}$  by Lemma 3.7.

If  $O \notin \hat{F}$ , then  $\frac{4}{3} + \frac{2x}{3} - 2a \geq (D \cdot C)_P \geq 2m + \tilde{m} + (\hat{D} \cdot \hat{C})_O > m + 3 - 3a$  by Theorem 3.4. Therefore, if  $O \notin \hat{F}$ , then  $a > \frac{5}{3} - \frac{2x}{3} \geq 1$ . But  $a \leq \frac{2}{3}$ . This shows that  $O = G \cap \hat{F} \cap \hat{C}$ . Then  $(\hat{D} \cdot \hat{C})_O > 1 - (2a + m - 1) - (3a + m + \tilde{m} - 2) = 4 - 5a - 2m - \tilde{m}$  by Theorem 3.4. Then  $\frac{4}{3} + \frac{2x}{3} - 2a \geq (D \cdot C)_P \geq 2m + \tilde{m} + (\hat{D} \cdot \hat{C})_O > 4 - 5a$ , which gives  $a > \frac{2}{3}$ .  $\square$

**Lemma 4.7.** *Suppose that  $C$  and  $Z$  are smooth at  $P$ ,  $(C \cdot Z)_P \leq 2$ , and  $a + b + m \leq 1 + \frac{x}{2}$ . Suppose also that  $a \leq \frac{1+x}{3}$ ,  $b \leq \frac{1+x}{3}$ ,  $(D \cdot C)_P \leq 1 + a - 2b$  and  $(D \cdot Z)_P \leq 1 + b - 2a$ . Then the log pair  $(S, aC + bZ + D)$  is log canonical at  $P$ .*

*Proof.* We have  $m \leq (D \cdot C)_P \leq 1 + a - 2b$  and  $m \leq (D \cdot Z)_P \leq 1 + b - 2a$ . Then  $m + \frac{a+b}{2} \leq 1$ .

Suppose that  $(S, aC + bZ + D)$  is not log canonical at  $P$ . Let us seek for a contradiction. We may assume that (3.5) is not log canonical at  $Q$ . Then  $Q \in \tilde{C} \cup \tilde{Z}$  by Lemma 3.9. Without loss of generality, we may assume that  $\tilde{C}$  contains  $Q$ . Then  $\tilde{Z}$  also contains  $Q$ . Indeed, if  $Q \notin \tilde{Z}$ , then  $1 + a - 2b \geq (D \cdot C)_P \geq m + (\tilde{D} \cdot \tilde{C})_Q > 2 - a - b$  by Theorem 3.4. But  $1 + b - 2a \geq 0$ . Thus, we have  $Q = G \cap \tilde{C} \cap \tilde{Z}$ , so that  $(C \cdot Z)_P = 2$ .

We may assume that (3.6) is not log canonical at  $O$ . Then  $O \in \hat{C} \cup \hat{Z}$  by Lemma 3.10. In particular, we have  $O \notin \hat{F}$ . Without loss of generality, we may assume that  $O \in \hat{C}$ . By Theorem 3.4, we have  $1 + a - 2b - m - \tilde{m} \geq (\hat{D} \cdot \hat{C})_O > 1 - (2a + 2b + m + \tilde{m} - 2)$ . This gives  $a > \frac{2}{3}$ , which is impossible, since  $a \leq 1 + \frac{x}{2} \leq \frac{2}{3}$ .  $\square$

**Lemma 4.8.** *Suppose that  $C$  and  $Z$  are smooth at  $P$ ,  $(C \cdot Z)_P \leq 2$ , and  $a + b + m \leq \frac{4}{3} + \frac{x}{6}$ . Suppose also that  $a \leq \frac{2}{3}$ ,  $b \leq \frac{2}{3}$ ,  $(D \cdot C)_P \leq \frac{2+x}{3} + a - 2b$  and  $(D \cdot Z)_P \leq \frac{2+x}{3} + b - 2a$ . Then the log pair  $(S, aC + bZ + D)$  is log canonical at  $P$ .*

*Proof.* We have  $m \leq (D \cdot C)_P \leq \frac{2+x}{3} + a - 2b$  and we have  $m \leq (D \cdot Z)_P \leq \frac{2+x}{3} + b - 2a$ . Then  $m + \frac{a+b}{2} \leq \frac{2+x}{3} \leq 1$ ,  $m + a + b \leq \frac{4}{3} + \frac{x}{6} \leq \frac{3}{2}$  and  $2a - b \leq 1$ .

Suppose that  $(S, aC + bZ + D)$  is not log canonical at  $P$ . Let us seek for a contradiction. We may assume that (3.5) is not log canonical at  $Q$ . Then  $Q \in \tilde{C} \cup \tilde{Z}$  by Lemma 3.9. Without loss of generality, we may assume that  $Q$  is contained in  $\tilde{C}$ . Then  $Q \in \tilde{C} \cap \tilde{Z}$ . Indeed, if  $\tilde{Z}$  does not contain  $Q$ , then  $\frac{2+x}{3} + a - 2b \geq m + (\tilde{D} \cdot \tilde{C})_Q > 2 - a - b$  by Theorem 3.4. The later inequality immediately leads to a contradiction, since  $2a - b \leq 1$ .

We may assume that (3.6) is not log canonical at  $O$ . Then  $O \in \hat{C} \cup \hat{Z}$  by Lemmas 3.10. In particular, we have  $O \notin \hat{F}$ . Without loss of generality, we may assume that  $O \in \hat{C}$ . Then  $\frac{2+x}{3} + a - 2b - m - \tilde{m} \geq (\hat{D} \cdot \hat{C})_O > 1 - (2a + 2b + m + \tilde{m} - 2)$  by Theorem 3.4. This gives  $a > \frac{7-x}{9}$ , which is impossible, since  $a \leq \frac{2}{3}$ .  $\square$

## 5. THE PROOF OF MAIN RESULT

Let  $S$  be a smooth del Pezzo surface such that  $K_S^2 = 1$ . Then  $|-2K_S|$  is base point free. It is well-known that the linear system  $|-2K_S|$  gives a double cover  $S \rightarrow \mathbb{P}(1, 1, 2)$ . This double cover induces an involution  $\tau \in \text{Aut}(S)$ .

Let  $C$  be an irreducible curve in  $S$  such that  $C^2 = -1$ . Then  $-K_S \cdot C = 1$  and  $C \cong \mathbb{P}^1$ . Put  $\tilde{C} = \tau(C)$ . Then  $\tilde{C}^2 = K_S \cdot \tilde{C} = -1$  and  $\tilde{C} \cong \mathbb{P}^1$ . Moreover, we have  $C + \tilde{C} \sim -2K_S$ . Furthermore, the irreducible curve  $\tilde{C}$  is uniquely determined by this rational equivalence. Since  $C \cdot (C + \tilde{C}) = -2K_S \cdot C = 2$  and  $C^2 = -1$ , we have  $C \cdot \tilde{C} = 3$ , so that  $1 \leq |C \cap \tilde{C}| \leq 3$ .

Fix  $\lambda \in \mathbb{Q}$ . Then  $-K_S + \lambda C$  is ample  $\iff -\frac{1}{3} < \lambda < 1$ . Indeed, we have

$$(5.1) \quad -K_S + \lambda C \sim_{\mathbb{Q}} \frac{1}{2}(C + \tilde{C}) + \lambda C = \left(\frac{1}{2} + \lambda\right)C + \frac{1}{2}\tilde{C} \sim_{\mathbb{Q}} (1 + 2\lambda)\left(-K_S - \frac{\lambda}{1 + 2\lambda}\tilde{C}\right).$$

On the other hand, we have  $(-K_S + \lambda C) \cdot C = 1 - \lambda$  and  $(-K_S + \lambda C) \cdot \tilde{C} = 1 - 3\lambda$ .

Note that Theorem 1.6 and (5.1) imply

**Corollary 5.1.** *Suppose that  $-\frac{1}{3} < \lambda < 1$ . If  $|C \cap \tilde{C}| \geq 2$ , then*

$$\alpha(S, -K_S + \lambda C) = \begin{cases} \min\left(\frac{\alpha(X)}{1 + 2\lambda}, 2\right) & \text{if } -\frac{1}{3} < \lambda < 0, \\ \min\left(\alpha(X), \frac{2}{1 + 2\lambda}\right) & \text{if } 0 \leq \lambda < 1. \end{cases}$$

Similarly, if  $|C \cap \tilde{C}| = 1$ , then

$$\alpha(S, -K_S + \lambda C) = \begin{cases} \min\left(\frac{\alpha(X)}{1 + 2\lambda}, \frac{4}{3 + 3\lambda}\right) & \text{if } -\frac{1}{3} < \lambda < 0, \\ \min\left(\alpha(X), \frac{4}{3 + 3\lambda}\right) & \text{if } 0 \leq \lambda < 1. \end{cases}$$

Now let us prove Theorem 1.6. Suppose that  $0 \leq \lambda < 1$ . Put

$$(5.2) \quad \mu = \begin{cases} \min\left(\alpha(S), \frac{2}{1 + 2\lambda}\right) & \text{when } |C \cap \tilde{C}| \geq 2, \\ \min\left(\alpha(S), \frac{4}{3 + 3\lambda}\right) & \text{when } |C \cap \tilde{C}| = 1. \end{cases}$$

**Lemma 5.2.** *One has  $\alpha(S, -K_S + \lambda C) \leq \mu$ .*

*Proof.* Since we have  $(\frac{1}{2} + \lambda)C + \frac{1}{2}\tilde{C} \sim_{\mathbb{Q}} -K_S + \lambda C$ , we see that  $\alpha(S, -K_S + \lambda C) \leq \frac{2}{1 + 2\lambda}$ . Similarly, we see that  $\alpha(S, -K_S + \lambda C) \leq \alpha(S)$ . If  $|C \cap \tilde{C}| = 1$ , then the log pair

$$\left(S, \frac{2 + 4\lambda}{3 + 3\lambda}C + \frac{2}{4 + 3\lambda}\tilde{C}\right)$$

is not Kawamata log terminal at the point  $C \cap \tilde{C}$ , so that  $\alpha(S, -K_S + \lambda C) \leq \frac{4}{3 + 3\lambda}$ .  $\square$

Thus, to complete the proof of Theorem 1.6, we have to show that  $\alpha(S, -K_S + \lambda C) \geq \mu$ . Suppose that  $\alpha(S, -K_S + \lambda C) < \mu$ . Let us seek for a contradiction.

Since  $\alpha(S, -K_S + \lambda C) < \mu$ , there exists an effective  $\mathbb{Q}$ -divisor  $D$  on  $S$  such that

$$D \sim_{\mathbb{Q}} -K_S + \lambda C,$$

and  $(S, \mu D)$  is not log canonical at some point  $P \in S$ .

By Lemma 3.2 and (5.1), we may assume that  $\text{Supp}(D)$  does not contain  $C$  or  $\tilde{C}$ . Indeed, one can check that the log pair  $(S, \mu(\frac{1}{2} + \lambda)C + \frac{\mu}{2}\tilde{C})$  is log canonical at  $P$ .

Let  $\mathcal{C}$  be a curve in the pencil  $|-K_S|$  that passes through  $P$ . Then  $\mathcal{C} + \lambda C \sim -K_S + \lambda C$ . Moreover, the curve  $\mathcal{C}$  is irreducible, and the log pair  $(S, \mu\mathcal{C} + \mu\lambda C)$  is log canonical at  $P$ . Thus, we may assume that  $\text{Supp}(D)$  does not contain  $C$  or  $\mathcal{C}$  by Lemma 3.2.

**Lemma 5.3.** *The curve  $\mathcal{C}$  is smooth at the point  $P$ .*

*Proof.* Suppose that  $\mathcal{C}$  is singular at  $P$ . If  $\mathcal{C} \not\subseteq \text{Supp}(D)$ , then Theorem 3.3 gives

$$1 + \lambda = \mathcal{C} \cdot (-K_S + \lambda C) = \mathcal{C} \cdot D \geq \text{mult}_P(\mathcal{C})\text{mult}_P(D) \geq 2\text{mult}_P(D) > \frac{2}{\mu},$$

which is impossible by (5.2). Thus, we have  $\mathcal{C} \subseteq \text{Supp}(D)$ . Then  $C \not\subseteq \text{Supp}(D)$ .

Write  $D = \epsilon\mathcal{C} + \Delta$ , where  $\epsilon$  is a positive rational number, and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor on the surface  $S$  whose support does not contain the curves  $\mathcal{C}$  and  $C$ . Then

$$1 - \lambda = C \cdot (-K_S + \lambda C) = C \cdot D = C \cdot (\epsilon\mathcal{C} + \Delta) = \epsilon + C \cdot \Delta \geq \epsilon,$$

so that  $\epsilon \leq 1 - \lambda$ . Similarly, we have

$$(5.3) \quad 1 + \lambda - \epsilon = \mathcal{C} \cdot \Delta \geq (\mathcal{C} \cdot \Delta)_P.$$

We claim that  $\lambda \leq \frac{1}{2}$ . Indeed, suppose that  $\lambda > \frac{1}{2}$ . Then it follows from (5.3) that

$$(\Delta \cdot \mathcal{C})_P \leq 1 + \lambda - \epsilon = \frac{1 + 2\lambda}{2} \left( \frac{4}{3} + \frac{\frac{4-4\lambda}{1+2\lambda}}{6} - \frac{2}{1+2\lambda}\epsilon \right).$$

Thus, we can apply Lemma 4.1 to the log pair  $(S, \frac{2}{1+2\lambda}D)$  with  $x = \frac{4-4\lambda}{1+2\lambda}$  and  $a = \frac{2}{1+2\lambda}\epsilon$ . This implies that  $(S, \frac{2}{1+2\lambda}D)$  is log canonical at  $P$ , which is impossible, because  $\mu \leq \frac{2}{1+2\lambda}$ .

If  $\mathcal{C}$  has a node at  $P$ , then we can apply Lemma 4.2 to  $(S, D)$  with  $x = 2\lambda$  and  $a = \epsilon$ . This implies that  $(S, D)$  is log canonical, which is absurd, since  $\mu \leq 1$ .

Therefore, the curve  $\mathcal{C}$  has an ordinary cusp at  $P$  and  $\lambda \leq \frac{1}{2}$ . Then  $\mu \leq \alpha(S) = \frac{5}{6}$ . Thus, we can apply Lemma 4.1 to the log pair  $(S, \frac{5}{6}D)$  with  $x = \frac{5}{3}\lambda$  and  $a = \frac{5}{6}\epsilon$ , since

$$(\Delta \cdot \mathcal{C})_P \leq \frac{6}{5} \left( \frac{5}{6} + \frac{5}{6}\lambda - \frac{5}{6}\epsilon \right).$$

This implies that  $(S, \frac{5}{6}D)$  is log canonical at  $P$ , which is impossible, since  $\mu \leq \frac{5}{6}$ .  $\square$

The next step in the proof of Theorem 1.6 is

**Lemma 5.4.** *The point  $P$  is not contained in the curve  $C$ .*

*Proof.* Suppose that  $P \in C$ . Let us seek for a contradiction. If  $C \not\subseteq \text{Supp}(D)$ , then

$$1 - \lambda = C \cdot (-K_S + \lambda C) = C \cdot D \geq \text{mult}_P(C)\text{mult}_P(D) \geq \text{mult}_P(D) > \frac{1}{\mu}$$

by Theorem 3.3. But (5.2) implies that  $\mu > \frac{1}{1-\lambda}$ , which is impossible, because  $\mu \leq 1$ . Therefore, we must have  $C \subseteq \text{Supp}(D)$ . Then  $\mathcal{C} \not\subseteq \text{Supp}(D)$  and also  $\tilde{C} \not\subseteq \text{Supp}(D)$ .

Write  $D = \epsilon C + \Delta$ , where  $\epsilon$  is a positive rational number, and  $\Delta$  is an effective divisor whose support does not contain  $\mathcal{C}$ ,  $C$  and  $\tilde{C}$ . Then  $1 + \lambda - \epsilon = \mathcal{C} \cdot \Delta \geq \text{mult}_P(\Delta)$ . Similarly, we have  $1 + 3\lambda - 3\epsilon = \tilde{C} \cdot \Delta \geq 0$ . Finally, we have  $1 - \lambda + \epsilon = C \cdot \Delta \geq (C \cdot \Delta)_P$ .

If  $\lambda \leq \frac{1}{2}$ , we can apply Lemma 4.3 to the log pair  $(S, D)$  with  $x = 2\lambda$  and  $a = \epsilon$ . This implies that  $(S, D)$  is log canonical, which is impossible since  $\mu \leq 1$ .

Therefore, we have  $\lambda > \frac{1}{2}$ . Since  $\epsilon \leq \frac{1}{3} + \lambda$ , we have  $\frac{2}{1+2\lambda}\epsilon \leq \frac{2}{1+2\lambda}(\frac{1}{3} + \lambda) = \frac{8}{9} - \frac{4-4\lambda}{18}$ . Since  $\epsilon + \text{mult}_P(\Delta) \leq 1 + \lambda$ , we have  $\frac{2}{1+2\lambda}\epsilon + \frac{2}{1+2\lambda}\text{mult}_P(\Delta) \leq \frac{2}{1+2\lambda}(1 + \lambda) = \frac{4}{3} + \frac{4-4\lambda}{6}$ . But

$$(\Delta \cdot C)_P \leq 1 - \lambda + \epsilon = \frac{1 + 2\lambda}{2} \left( \frac{\frac{4-4\lambda}{1+2\lambda}}{2} + \frac{2}{1+2\lambda}\epsilon \right).$$

Thus, we can apply Lemma 4.4 to the log pair  $(S, \frac{2}{1+2\lambda}D)$  with  $x = \frac{4-4\lambda}{1+2\lambda}$  and  $a = \frac{2}{1+2\lambda}\epsilon$ . This implies that  $(S, \frac{2}{1+2\lambda}D)$  is log canonical at  $P$ , which is impossible, since  $\mu \leq \frac{2}{1+2\lambda}$ .  $\square$

Let  $h: S \rightarrow \bar{S}$  be the contraction of the curve  $C$ . Put  $\bar{D} = h(D)$ . Then  $\bar{D} \sim_{\mathbb{Q}} -K_{\bar{S}}$ . Moreover, it follows from Lemma 5.4 that  $(\bar{S}, \mu\bar{D})$  is not log canonical at the point  $h(P)$ .

By construction, the surface  $\bar{S}$  is a smooth del Pezzo surface such that  $K_{\bar{S}}^2 = K_S^2 + 1 = 2$ . Then  $|-K_{\bar{S}}|$  gives a double cover  $\pi: \bar{S} \rightarrow \mathbb{P}^2$  branched in a smooth quartic curve  $R_4 \subset \mathbb{P}^2$ . By Lemma 3.6, there exists a unique curve  $\bar{Z} \in |-K_{\bar{S}}|$  such that  $\bar{Z}$  is singular at  $h(P)$ . Moreover, the log pair  $(\bar{S}, \bar{Z})$  is not log canonical at the point  $h(P)$  by [4, Theorem 1.12]. Note that  $\pi(\bar{Z})$  is the line in  $\mathbb{P}^2$  that is tangent to the curve  $R_4$  at the point  $\pi \circ h(P)$ .

Let  $Z$  be the proper transform of the curve  $\bar{Z}$  on the surface  $S$ . Then  $h(C) \not\subseteq \bar{Z}$ . Indeed, if  $h(C)$  is contained in  $\bar{Z}$ , then  $Z \sim -K_S$ , which is impossible by Lemma 5.3. Thus, we see that  $C \cap Z = \emptyset$ . Then  $Z \sim -K_S + C$ .

**Lemma 5.5.** *The curve  $Z$  is reducible.*

*Proof.* Suppose that  $Z$  is irreducible. Then  $Z$  has an ordinary node or ordinary cusp at  $P$ . In fact, if  $Z \not\subseteq \text{Supp}(D)$ , then  $2 = Z \cdot D > \frac{2}{\mu}$  by Theorem 3.3, which contradicts to (5.2). Therefore, we have  $Z \subseteq \text{Supp}(D)$ . Put  $\tilde{Z} = \tau(Z)$ . Then  $Z + \tilde{Z} \sim -4K_S$  and

$$\frac{3\lambda + 1}{4}Z + \frac{1 - \lambda}{4}\tilde{Z} \sim_{\mathbb{Q}} \frac{1 - \lambda}{4}(Z + \tilde{Z}) + \lambda Z \sim_{\mathbb{Q}} -K_S + \lambda C.$$

Furthermore, one can show (using Definition 3.1) that the log pair

$$\left( S, \mu \frac{3\lambda + 1}{4}Z + \mu \frac{1 - \lambda}{4}\tilde{Z} \right)$$

is log canonical at  $P$ . Hence, we may assume that  $\tilde{Z} \not\subseteq \text{Supp}(D)$  by Lemma 3.2.

Write  $D = \epsilon Z + \Delta$ , where  $\epsilon$  is a positive rational number, and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor on the surface  $S$  whose support does not contain  $Z$  and  $\tilde{Z}$ . Then  $2 + 4\lambda - 6\epsilon = \tilde{Z} \cdot \Delta \geq 0$ . Thus, we have  $\epsilon \leq \frac{1+2\lambda}{3}$ . Finally, we have

$$2 - 2\epsilon = Z \cdot \Delta \geq (Z \cdot \Delta)_P.$$

Therefore, if  $\lambda \leq \frac{1}{2}$ , then we can apply Lemma 4.5 to  $(S, D)$  with  $x = 2\lambda$  and  $a = \epsilon$ . This implies that  $(S, D)$  is log canonical at  $P$ . But  $\mu \leq 1$ . Thus, we have  $\lambda > \frac{1}{2}$ .

We have  $\mu \leq \frac{2}{1+2\lambda}$ . Then  $(S, \frac{2}{1+2\lambda}D)$  is not log canonical at  $P$ . We have  $\frac{2}{1+2\lambda}\epsilon \leq \frac{2}{3}$ . Thus, we can apply Lemma 4.6 to  $(S, \frac{2}{1+2\lambda}D)$  with  $x = \frac{4-4\lambda}{1+2\lambda}$  and  $a = \frac{2}{1+2\lambda}\epsilon$ , because

$$(\Delta \cdot Z)_P \leq \frac{1+2\lambda}{2} \left( \frac{4}{3} + \frac{2\frac{4-4\lambda}{1+2\lambda}}{3} - 2\frac{2}{1+2\lambda}\epsilon \right) = 2 - 2\epsilon.$$

This implies that  $(S, \frac{2}{1+2\lambda}D)$  is log canonical at  $P$ , which is absurd, since  $\mu \leq \frac{2}{1+2\lambda}$ .  $\square$

Since  $Z$  is reducible,  $Z = Z_1 + Z_2$ , where  $Z_1$  and  $Z_2$  are smooth irreducible curves. Then  $Z_1^2 = Z_2^2 = -1$  and  $Z_1 \cdot Z_2 = 2$ . Moreover, we have  $P \in Z_1 \cap Z_2$  and  $(Z_1 \cdot Z_2)_P \leq 2$ . Furthermore, we have  $Z_1 \cap C = \emptyset$  and  $Z_2 \cap C = \emptyset$ .

We have  $Z_1 \subseteq \text{Supp}(D)$  and  $Z_2 \subseteq \text{Supp}(D)$ . Indeed, if  $Z_1 \not\subseteq \text{Supp}(D)$ , then

$$1 = Z_1 \cdot (-K_S + \lambda C) = Z_1 \cdot D \geq \text{mult}_P(Z_1) \text{mult}_P(D) \geq \text{mult}_P(D) > \frac{1}{\mu} \geq 1$$

by Theorem 3.3. This shows that  $Z_1 \subseteq \text{Supp}(D)$ . Similarly, we have  $Z_2 \subseteq \text{Supp}(D)$ . But

$$(1 - \lambda)\mathcal{C} + \lambda(Z_1 + Z_2) \sim_{\mathbb{Q}} -K_S + \lambda C.$$

On the other hand, the log pair  $(S, \mu(1 - \lambda)\mathcal{C} + \mu\lambda(Z_1 + Z_2))$  is log canonical at  $P$ . Therefore, we may assume that  $\mathcal{C} \not\subseteq \text{Supp}(D)$  by Lemma 3.2.

Put  $\tilde{Z}_1 = \tau(Z_1)$  and put  $\tilde{Z}_2 = \tau(Z_2)$ . Then  $Z_1 + \tilde{Z}_1 \sim -2K_S$  and  $Z_2 + \tilde{Z}_2 \sim -2K_S$ . This gives  $\mathcal{C} \cdot Z_1 = \mathcal{C} \cdot Z_2 = 1$ ,  $Z_1 \cdot \tilde{Z}_1 = Z_2 \cdot \tilde{Z}_2 = 3$ ,  $Z_1 \cdot \tilde{Z}_2 = Z_2 \cdot \tilde{Z}_1 = 0$ ,  $\tilde{Z}_1 \cdot C = \tilde{Z}_2 \cdot C = 2$ . Moreover, we have  $Z_1 + Z_2 \sim -K_S + C$ . Then

$$\frac{1+\lambda}{2}Z_1 + \lambda Z_2 + \frac{1-\lambda}{2}\tilde{Z}_1 \sim_{\mathbb{Q}} \frac{1-\lambda}{2}(Z_1 + \tilde{Z}_1) + \lambda(Z_1 + Z_2) \sim_{\mathbb{Q}} -K_S + \lambda C$$

Note that  $P \notin \tilde{Z}_1$ , because  $P \in Z_2$  and  $\tilde{Z}_1 \cdot Z_2 = 0$ . Using this, we see that the log pair

$$\left( S, \mu \frac{1+\lambda}{2}Z_1 + \mu\lambda Z_2 + \mu \frac{1-\lambda}{2}\tilde{Z}_1 \right)$$

is log canonical at the point  $P$ . Hence, we may assume that  $\tilde{Z}_1 \not\subseteq \text{Supp}(D)$  by Lemma 3.2. Similarly, we may assume that  $\tilde{Z}_2 \not\subseteq \text{Supp}(D)$  using Lemma 3.2 one more time.

Now let us write  $D = \epsilon_1 Z_1 + \epsilon_2 Z_2 + \Delta$ , where  $\epsilon_1$  and  $\epsilon_2$  are positive rational numbers, and  $\Delta$  is an effective divisor whose support does not contain  $Z_1$  and  $Z_2$ . Then

$$1 + \lambda - \epsilon_1 - \epsilon_2 = \mathcal{C} \cdot \Delta \geq \text{mult}_P(\Delta).$$

This gives  $\epsilon_1 + \epsilon_2 + \text{mult}_P(\Delta) \leq 1 + \lambda$ . We also have  $\epsilon_1 \leq \frac{1+2\lambda}{3}$ , since

$$1 + 2\lambda - 3\epsilon_1 = \tilde{Z}_1 \cdot \Delta \geq 0.$$

Similarly, see that  $\epsilon_2 \leq \frac{1+2\lambda}{3}$ . Moreover, we have

$$1 + \epsilon_1 - 2\epsilon_2 = Z_1 \cdot \Delta \geq (Z_1 \cdot \Delta)_P.$$

Finally, we have

$$1 + \epsilon_2 - 2\epsilon_1 = Z_2 \cdot \Delta \geq (Z_2 \cdot \Delta)_P.$$



Thus, if  $\lambda \leq \frac{1}{2}$ , then we can apply Lemma 4.7 to  $(S, D)$  with  $x = 2\lambda$ ,  $a = \epsilon_1$  and  $b = \epsilon_1$ . This implies that  $(S, D)$  is log canonical at  $P$ , which is absurd. Hence, we have  $\lambda > \frac{1}{2}$ .

Since  $\lambda > \frac{1}{2}$ , we have  $\mu \leq \frac{2}{1+2\lambda}$ . Then the log pair  $(S, \frac{2}{1+2\lambda}D)$  is not log canonical at  $P$ . On the other hand, we have  $\frac{2}{1+2\lambda}\epsilon_1 \leq \frac{2}{3}$  and  $\frac{2}{1+2\lambda}\epsilon_2 \leq \frac{2}{3}$ . We also have

$$\frac{2}{1+2\lambda}\epsilon_1 + \frac{2}{1+2\lambda}\epsilon_2 + \frac{2}{1+2\lambda}\text{mult}_P(\Delta) \leq \frac{2}{1+2\lambda}(1+\lambda) = \frac{2}{1+2\lambda} + \lambda \frac{2}{1+2\lambda} = \frac{4}{3} + \frac{4-4\lambda}{6},$$

Moreover, we have

$$(\Delta \cdot Z_1)_P \leq 1 + \epsilon_1 - 2\epsilon_2 = \frac{1+2\lambda}{2} \left( \frac{2}{3} + \frac{\frac{4-4\lambda}{1+2\lambda}}{3} + \frac{2}{1+2\lambda}\epsilon_1 - 2\frac{2}{1+2\lambda}\epsilon_2 \right).$$

Furthermore, we also have

$$(\Delta \cdot Z_2)_P \leq 1 + \epsilon_1 - 2\epsilon_2 = \frac{1+2\lambda}{2} \left( \frac{2}{3} + \frac{\frac{4-4\lambda}{1+2\lambda}}{3} + \frac{2}{1+2\lambda}\epsilon_2 - 2\frac{2}{1+2\lambda}\epsilon_1 \right).$$

Thus, we can apply Lemma 4.8 to  $(S, \frac{2}{1+2\lambda}D)$  with  $x = \frac{4-4\lambda}{1+2\lambda}$ ,  $a = \frac{2}{1+2\lambda}\epsilon_1$  and  $b = \frac{2}{1+2\lambda}\epsilon_2$ . This implies that  $(S, \frac{2}{1+2\lambda}D)$  is log canonical at  $P$ , which is absurd.

The obtained contradiction completes the proof of Theorem 1.6.

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